On the spin-liquid phase of one dimensional spin-1 bosons

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Abstract. We consider a model of one dimensional spin-1 bosons with repulsive density-density interactions and antiferromagnetic exchange. We show that the low energy effective field theory is given by a spin-charge separated theory of a Tomonaga-Luttinger Hamiltonian and the O(3) nonlinear sigma model describing collective charge and spin excitations respectively. At a particular ratio of the density-density to spin-spin interaction the model is integrable, and we use the exact solutions to provide an independent derivation of the low energy effective theory. The system is in a superfluid phase made of singlet pairs of bosons, and we calculate the long-distance asymptotics of certain correlation functions.

1. Introduction

Spinor Bose gases exhibit a much richer variety of interesting macroscopic quantum phenomena than spinless bosons. A mean field analysis of three-dimensional spin-1 Bose-condensed gases [1, 2] and exact diagonalization of the spin Hamiltonian for this system [3] showed that the ground state can be either ferromagnetic or spin-singlet, depending on the sign of the coupling constant for the exchange part of the interaction. In an optical lattice Mott phases occur, featuring an interesting behaviour in the magnetic sector [4, 5]. Experimentally, the ground state of such gases, its magnetic properties and low temperature dynamical properties have been studied for ^{23}Na, which has a spin-singlet ground state [6], and for ^{87}Rb where the ground state is ferromagnetic [7–9]. The creation of a one-dimensional (1D) regime for quantum gases by tightly confining the motion of particles in two directions offered new possibilities for studying macroscopic quantum effects, and a number of advanced experiments have been done for spinless 1D bosons [10, 11].

In this paper we consider a 1D liquid of spin-1 bosons with a short-range interaction containing a repulsive density-density term and a magnetic exchange interaction. We demonstrate that for the case of antiferromagnetic exchange the low energy effective action decouples into two parts: the standard U(1) Tomonaga-Luttinger action describing density fluctuations, and the action of the O(3) nonlinear sigma model describing the spin sector. The latter model has a singlet ground state separated from the first excited triplet by a gap. Therefore the system of spin-1 bosons interacting via antiferromagnetic exchange constitutes a perfect spin liquid where spin-spin correlations decay exponentially at large distances. We also identify the "order parameter field", which has correlations exhibiting a power law decay at T=0. This field consists of pairs of bosons

$$\Delta = \Psi_1\Psi_{-1} - 1/2\Psi_0^2. \tag{1}$$

Hence the fluctuation superfluidity of spin-1 bosons is really a superfluidity of pairs.

We derive our results by two means. For weak interactions we perform a semiclassical analysis of the bosonic path integral. For a special point where the coupling constants for the density-density and exchange interaction are equal to each other we study the Bethe ansatz solution. The latter result is of a certain interest since it turns out that the integrable model of spin-1 bosons may serve as an integrable regulator for the O(3) nonlinear sigma model.

2. Low-energy effective action. Semiclassical derivation.

Our starting point is a three-component Bose gas with Hamiltonian density

$$\mathcal{H} = \frac{1}{2m}\partial_x\mathbf{\Psi}^\dagger\partial_x\mathbf{\Psi} - \mu\mathbf{\Psi}^\dagger\mathbf{\Psi} + \frac{g_0}{2}\left[\mathbf{\Psi}^\dagger\mathbf{\Psi}\right]^2 + \frac{g_1}{2}\left[\mathbf{\Psi}^\dagger\boldsymbol{S}^\alpha\mathbf{\Psi}\right]\left[\mathbf{\Psi}^\dagger\boldsymbol{S}^\alpha\mathbf{\Psi}\right]. \tag{2}$$

Here $\mathbf{\Psi}(x) = (\Psi_1(x), \Psi_0(x), \Psi_{-1}(x))$ is a three-component Bose field and $\boldsymbol{S}^\alpha$ are spin-1 matrices. We now introduce a number-phase representation

$$\Psi_\sigma(x) = \sqrt{\rho(x)}n_\sigma(x)e^{i\phi_\sigma(x)}, \tag{3}$$

with a real unit vector field $n_\sigma(x)$

$$\sum_\sigma n_\sigma^2 = 1. \tag{4}$$

In the number-phase representation the Hamiltonian density takes the form $\mathcal{H} = \mathcal{H}_0 + V$, where

$$\mathcal{H}_0 = \frac{1}{8m\rho}(\partial_x\rho)^2 + \frac{\rho}{2m}\sum_\sigma n_\sigma^2(\partial_x\phi_\sigma)^2 + (\partial_x n_\sigma)^2, \tag{5}$$

$$\begin{aligned} V \;=\; & -\mu\rho + \frac{g_0}{2}\rho^2 + \frac{g_1}{2}\rho^2\Big[(n_1^2 - n_{-1}^2)^2 + 2n_0^2(n_1^2 + n_{-1}^2) \\ & + 4n_0^2 n_1 n_{-1}\cos(2\phi_0 - \phi_1 - \phi_{-1})\Big]. \end{aligned} \tag{6}$$

In order to minimize the potential we eliminate n_0 via the constraint (4) and define

$$\phi_\pm = \frac{\phi_1 \pm \phi_{-1}}{2}\ , \qquad \tilde{\phi} = 2\phi_0 - \phi_1 - \phi_{-1}, \qquad n_\pm = n_1 \pm n_{-1}. \tag{7}$$

The new phase fields are taken to have ranges

$$0 \le \phi_+ < \pi\ , \quad 0 \le \phi_- < 2\pi\ . \tag{8}$$

In the new variables the potential takes the form

$$\begin{aligned} V = & \frac{g_1}{2}\rho^2\left[n_+^2 n_-^2 + (2 - n_+^2 - n_-^2)(n_+^2 - \frac{n_+^2 - n_-^2}{2}[1 - \cos\tilde{\phi}])\right] \\ & - \mu\rho + \frac{g_0}{2}\rho^2. \end{aligned} \tag{9}$$

Minimizing the potential (9) gives the solutions (for $g_1 > 0$)

$$A: \quad \rho = \frac{\mu}{g_0} \equiv \rho_0\ , \quad \tilde{\phi} = 0\ , \quad n_+ = 0. \tag{10}$$

$$B: \quad \rho = \frac{\mu}{g_0} \equiv \rho_0\ , \quad \tilde{\phi} = \pi\ , \quad n_- = 0. \tag{11}$$

From now on we concentrate on solution A. Expanding the potential around the minimum gives

$$V = \frac{g_0}{2}(\rho - \rho_0)^2 + \frac{g_1\rho_0^2}{2}\left[2n_+^2 + \frac{1}{2}n_-^2(1 - \frac{n_-^2}{2})(\tilde{\phi})^2\right] + \dots \tag{12}$$

where the dots stand for terms of higher order in powers of $n_\pm, \tilde{\phi}$. Carrying out the analogous expansion for the kinetic energy part (5) of the Hamiltonian density gives

$$\begin{aligned} \mathcal{H}_0 = & \frac{\rho_0}{2m}\left[(\partial_x\phi_+)^2 + \frac{n_-^2}{2}(\partial_x\phi_-)^2 + (\partial_x n_-)^2\frac{1}{2 - (n_-)^2}\right] \\ & + \frac{1}{8m\rho_0}(\partial_x\rho)^2 + \dots\ . \end{aligned} \tag{13}$$

Putting everything together we arrive at the following expression for the action

$$
\begin{aligned}
S &= \int dtdx \left[i\mathbf{\Psi}^\dagger \partial_t \mathbf{\Psi} - \mathcal{H} \right] \\
&= \int dtdx \left[-\rho \partial_t \phi_+ - \rho_0 n_+ n_- \partial_t \phi_- - \frac{\rho_0}{2} \tilde{\phi} n_- \partial_t n_- + \ldots - \mathcal{H} \right] \qquad (14)
\end{aligned}
$$

We observe that ρ, n_+ and $\tilde{\phi}$ correspond to massive degrees of freedom. In the next step we integrate these out, which in our quadratic approximation can be done by "completing the squares". At energies well below the gaps for excitations in the ρ n_+ and $\tilde{\phi}$ sectors, retardation effects can be neglected and the low-energy effective action remains local in time. Integrating over $(\rho - \rho_0)$ generates a contribution

$$
\frac{1}{2g_0} (\partial_t \phi_+)^2, \qquad (15)
$$

to the action, while integrating out n_+ and $\tilde{\phi}$ generate respectively

$$
\frac{1}{4g_1} n_-^2 (\partial_t \phi_-)^2 \qquad (16)
$$

and

$$
\frac{1}{4g_1} (\partial_t n_-)^2 \frac{1}{1 - \frac{n_-^2}{2}}. \qquad (17)
$$

Finally we parametrize

$$
n_- = \sqrt{2} \sin \theta, \qquad (18)
$$

and arrive at the following form for the low-energy effective action $S = S_0 + S_{nl\sigma m}$:

$$
S_0 = \int dtdx \left[\frac{1}{2g_0} (\partial_t \phi_+)^2 - \frac{\rho_0}{2m} (\partial_x \phi_+)^2 \right]. \qquad (19)
$$

$$
\begin{aligned}
S_{nl\sigma m} = \int dtdx \Big\{ &\frac{1}{2g_1} \left((\partial_t \theta)^2 + \sin^2 \theta (\partial_t \phi_-)^2 \right) \\
&- \frac{\rho_0}{2m} \left[(\partial_x \theta)^2 + \sin^2 \theta (\partial_x \phi_-)^2 \right] \Big\}. \qquad (20)
\end{aligned}
$$

The action S_0 describes a free massless boson whereas $S_{nl\sigma m}$ is a parametrization for the O(3) nonlinear sigma model. Indeed, defining a three-dimensional unit vector field $\boldsymbol{m}^2 = 1$ by

$$
\boldsymbol{m} = \begin{pmatrix} \sin\theta \cos\phi_- \\ \sin\theta \sin\phi_- \\ \cos\theta \end{pmatrix}, \qquad (21)
$$

we obtain

$$
S_{nl\sigma m} = \frac{1}{2g_1} \int dtdx \left[(\partial_t \boldsymbol{m})^2 - v_s^2 (\partial_x \boldsymbol{m})^2 \right], \qquad (22)
$$

where the spin velocity is given by

$$
v_s = \sqrt{\frac{\rho_0 g_1}{m}}. \qquad (23)
$$

The action in the ϕ_+ sector can be brought to a more standard Luttinger liquid form

$$S_0 = \frac{K_c}{2\pi} \int dtdx \left[\frac{1}{v_c} (\partial_t \phi_+)^2 - v_c (\partial_x \phi_+)^2 \right], \tag{24}$$

where the charge velocity v_c and Luttinger parameter K_c are given by

$$v_c = \sqrt{\frac{\rho_0 g_0}{m}}, \quad K_c = \pi \sqrt{\frac{\rho_0}{m g_0}}. \tag{25}$$

We note that for equal coupling constants $g_0 = g_1$ the spin and charge velocities coincide, i.e. $v_s = v_c$.

2.1. Integration Measure

In the above derivation we have disregarded the integration measure in the path integral of our bosonic theory. We still have to verify that it correctly produces the integration measure for the nonlinear sigma model (20). We start by considering the Jacobian of the number-phase parametrization

$$\det \frac{\partial(\Psi_\sigma^\dagger, \Psi_\sigma)}{\partial(\rho, n_1, n_{-1}, \phi_\sigma)} = -4 i n_1 n_{-1} \rho^2. \tag{26}$$

Changing variables to $n_\pm$ gives a factor

$$\frac{1}{2}(n_-^2 - n_+^2) dn_+ dn_- \tag{27}$$

in the integration measure. Expanding n_+ around the minimum of the potential V leaves

$$n_-^2 dn_- = 2^{3/2} \sin^2\theta \cos\theta d\theta \tag{28}$$

The integration measure then looks:

$$\sqrt{2}\ d\rho\ d\phi_+\ d\phi_-\ dn_+\ d\theta\ d\tilde{\phi}\ \sin^2\theta \cos\theta. \tag{29}$$

When integrating out the $\tilde{\phi}$ field we wish to change variables to

$$\tilde{\phi}\ n_- \sqrt{1 - \frac{n_-^2}{2}} = \hat{\phi}. \tag{30}$$

In terms of the θ-parametrization (18) this gives

$$d\tilde{\phi} = \frac{d\hat{\phi}}{\sqrt{2}\sin\theta\cos\theta}, \tag{31}$$

which turns the measure into

$$d\rho\ dn_+\ d\hat{\phi}\ d\phi_+ \left[\sin\theta d\theta\ d\phi_- \right]. \tag{32}$$

The θ, ϕ_- piece is indeed the correct integration measure for the nonlinear sigma model.

We conclude that at low energies the spinor-Bose Hamiltonian (2) is described by a spin-charge separated theory of a free boson (24) describing collective superfluid pairing fluctuations and the O(3) nonlinear sigma model (22) describing spin excitations.

3. Bethe ansatz solution

We now turn to an entirely different analysis of the model (2), which will lead to the same conclusion. The first quantized form of the Hamiltonian (2) is

$$H = -\frac{1}{2m}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \frac{1}{2}\sum_{i\neq j}\delta(x_i - x_j)[g_0 + g_1\mathbf{S}_i\mathbf{S}_j], \tag{33}$$

where $\mathbf{S}$ are spin $S = 1$ operators. It was shown in Ref. [12] that the model (33) is integrable along the line $g_1 = g_0$. A simple understanding of certain features of the exact solution can be gained by considering the case of two bosons in the strong coupling limit (in which we neglect the kinetic energy). The potential energy of the pair of bosons at the same position is obtained by adding two spins $S = 1$ to form total angular momentum J:

$$E_J = g_0 + g_1[\frac{1}{2}J(J+1) - S(S+1)], \quad S = 1,\ J = 0, 1, 2. \tag{34}$$

The singlet $J = 0$ is always lower in energy than the triplet and quintet states and at $g_1 > g_0/2$ its energy becomes negative. In this regime the formation of $J = 0$ bound states becomes energetically favourable compared to having scattering states of two bosons. As a result the ground state is made of paired bosons. The integrable case $g_0 = g_1$ lies well within the parameter region where this occurs.

Let us now return to the Bethe ansatz analysis. According to Ref. [12] the Bethe ansatz equations are given by

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}k_i\mathrm{L}} &= -\prod_{j=1}^{N} e_4(k_i - k_j)\prod_{a=1}^{M} e_{-2}(k_i - \lambda_a)\ , \\
-1 &= \prod_{j=1}^{N} e_{-2}(\lambda_a - k_j)\prod_{b=1}^{M} e_2(\lambda_a - \lambda_b)\ , \\
E &= \frac{1}{2m}\sum_{i=1}^{N} k_i^2\ , \quad P = \sum_{i=1}^{N} k_i\ ,
\end{aligned} \tag{35}$$

where N is the total number of particles, $M = (N - S^z)$, E and P are respectively the energy and momentum and

$$e_n(x) = \frac{x + \mathrm{i}cn/2}{x - \mathrm{i}cn/2}. \tag{36}$$

The parameter c is related to the couplings $g_{0,1}$ by

$$c = mg_0 = mg_1. \tag{37}$$

In order to proceed, we now assume that the system is at a finite temperature $T \ll \mu$, where the chemical potential μ is the largest energy scale in the problem. As was shown in Ref. [12], in this case [13] only three types of (string) solutions of the Bethe ansatz equations (35) contribute to the thermodynamics of the model

- Bound states of k's and λ's (k-Λ strings [14–16]):
$$\begin{aligned} k_i^{\pm} &= \Lambda_i \pm \mathrm{i}c/2 + \gamma_i^{\pm}\ , \\ \lambda_i^{\pm} &= \Lambda_i \pm \mathrm{i}c/2 + \delta_i^{\pm}\ , \end{aligned} \tag{38}$$
where the string centres Λ_i are real, $\gamma, \delta \propto \exp(-\text{const } L)$. The zero temperature ground state is formed by a partially filled Fermi sea of these bound states.
- Real k's.
- λ-strings [14–17]:
$$\lambda_a^{(n,j)} = \lambda_a^{(n)} + \mathrm{i}c[(n+1)/2 - j] + O(\exp(-L\mathcal{N})), \quad j = 1, \ldots n\ , \tag{39}$$
where the string centres $\lambda_a^{(n)} \in \mathbb{R}$ and $\mathcal{N}$ is related to the number of excitations in this sector.

Substituting (38) into (35) and dropping the exponentially small corrections γ, δ we obtain the following set of equations
$$\mathrm{e}^{2\mathrm{i}L\Lambda_\mathrm{i}} = \prod_{j=1}^{N_b} \mathcal{E}(\Lambda_i - \Lambda_j) \prod_{p=1}^{N_r} \mathcal{P}(\Lambda_i - k_p)\ , \tag{40}$$
$$\mathrm{e}^{\mathrm{i}Lk_\mathrm{i}} = \prod_{j=1}^{N_r} e_4(k_i - k_j) \prod_{p=1}^{N_p} \mathcal{P}(k_i - \Lambda_p) \prod_{a=1}^{M_r} e_{-2}(k_i - \lambda_a)\ , \tag{41}$$
$$-1 = \prod_{j=1}^{N_r} e_{-2}(\lambda_a - k_j) \prod_{b=1}^{M_r} e_2(\lambda_a - \lambda_b)\ , \tag{42}$$
$$E = \frac{1}{2m}\left[\sum_{i=1}^{N_b}(2\Lambda_i^2 - c^2/2) + \sum_{i=1}^{N_r} k_i^2\right]\ , \quad P = \sum_{i=1}^{N_b} 2\Lambda_i + \sum_{p=1}^{N_r} k_p. \tag{43}$$
Here $N = N_r + 2N_b$, $S^z = N_r - M_r$ and
$$\mathcal{E}(x) = e_6(x)e_4(x)e_{-2}(x), \quad \mathcal{P}(x) = e_5(x)e_{-1}(x).$$
We now take the logarithm of the Bethe ansatz equations for the k-Λ strings forming the ground state and then express them in terms of a counting function [16, 18]
$$\begin{aligned} y(\Lambda) = 2\Lambda &+ \frac{1}{L}\sum_{j=1}^{N_b} \theta\Big(\frac{\Lambda - \Lambda_j}{3c}\Big) + \theta\Big(\frac{\Lambda - \Lambda_j}{2c}\Big) - \theta\Big(\frac{\Lambda - \Lambda_j}{c}\Big) \\ &+ \frac{1}{L}\sum_{p=1}^{N_r} \theta\Big(2\frac{\Lambda - k_p}{5c}\Big) - \theta\Big(2\frac{\Lambda - k_p}{c}\Big). \end{aligned} \tag{44}$$
Here $\theta(x) = 2\arctan(x)$. In terms of the counting function the Bethe ansatz equations (40) read
$$y(\Lambda_j) = \frac{2\pi I_j}{L}\ , \tag{45}$$
where I_j are integer (for N_b odd) of half-odd integer (for N_b even) numbers. We note that the momentum is expressed in terms of the I_j as
$$P = \frac{2\pi}{L}\sum_{i=1}^{N_b} I_j + \sum_{p=1}^{N_r} k_p. \tag{46}$$

As shown in [12] the ground state of the system corresponds to choosing $N_r = M_r = 0$, $2N_b = N$ and

$$I_j = -\frac{N_b+1}{2} + j \ , \quad j = 1, \dots N_b. \tag{47}$$

This immediately tells us that the "Fermi momentum" is

$$k_F = \frac{2\pi}{L}(I_{N_b} + \frac{1}{2}) = \frac{\pi N}{2L} = \frac{\pi}{2}\rho_0. \tag{48}$$

We are interested in the limit

$$L \to \infty \ , \quad N_b \to \infty \ , \quad \frac{2N_b}{L} = \rho_0 = \text{ const.} \tag{49}$$

In this limit we can turn the sum over j in (44) into an integral by means of the Euler-Maclaurin sum formula, resulting in

$$\begin{aligned} y(\Lambda) = 2\Lambda + \int_{-\infty}^{\infty} d\Lambda' \left[\theta\Big(\frac{\Lambda-\Lambda'}{3c}\Big) + \theta\Big(\frac{\Lambda-\Lambda'}{2c}\Big) - \theta\Big(\frac{\Lambda-\Lambda'}{c}\Big)\right] \sigma'_p(\Lambda') \\ + \frac{1}{L}\sum_{p=1}^{N_r} \theta\Big(2\frac{\Lambda-k_p}{5c}\Big) - \theta\Big(2\frac{\Lambda-k_p}{c}\Big). \end{aligned} \tag{50}$$

Here $\sigma'_{p,h}(\Lambda)$ is are the root densities of "particles" and "holes" for k-Λ strings. They are related to the counting function by

$$\frac{dy(\Lambda)}{d\Lambda} = 2\pi\left[\sigma'_p(\Lambda) + \sigma'_h(\Lambda)\right]. \tag{51}$$

Taking the derivative of (50) we obtain an integral equation

$$\sigma'_p(\Lambda) + \sigma'_h(\Lambda) = \frac{1}{\pi} + (a_6 + a_4 - a_2) * \sigma'_p\Big|_\Lambda + \frac{1}{L}\sum_{p=1}^{N_r} a_5(\Lambda - k_p) - a_1(\Lambda - k_p). \tag{52}$$

Here $*$ denotes a convolution

$$f * g\Big|_x = \int_{-\infty}^{\infty} dy \ f(x-y)g(y). \tag{53}$$

and the integral kernels are given by

$$a_n(x) = \frac{1}{2\pi}\frac{nc}{x^2 + (nc/2)^2}. \tag{54}$$

Equation (52) can be recast in the form

$$(I + a_2) * (I - a_4) * \sigma'_p\Big|_\Lambda = \frac{1}{\pi} - \sigma'_h(\Lambda) + \frac{1}{L}\sum_{p=1}^{N_r} a_5(\Lambda - k_p) - a_1(\Lambda - k_p). \tag{55}$$

In order to derive the Bethe ansatz equations (41) over the ground state of the system we need to express $\prod_{p=1}^{N_b} \mathcal{P}(k_i - \Lambda_p)$ in terms of the root densities for k-Λ strings. We have

$$\begin{aligned} \prod_{p=1}^{N_b} \mathcal{P}(k - \Lambda_p) &= \exp\left(-i\sum_{p=1}^{N_b} \theta\Big(2\frac{k-\Lambda_p}{5c}\Big) - \theta\Big(2\frac{k-\Lambda_p}{c}\Big)\right) \\ &= \exp\left(-iL\int_{-\infty}^{\infty} d\Lambda \sigma'_p(\Lambda)\left[\theta\Big(2\frac{k-\Lambda}{5c}\Big) - \theta\Big(2\frac{k-\Lambda}{c}\Big)\right]\right) \\ &\equiv \exp\left(2\pi i L f(k)\right). \end{aligned} \tag{56}$$

Taking derivatives we have

$$f'(k) = (a_1 - a_5) * \sigma_p\Big|_k = a_1 * (I - a_4) * \sigma_p\Big|_k. \tag{57}$$

On the other hand Eq. (55) implies that

$$a_1 * (I - a_4) * \sigma_p\Big|_k = \frac{1}{2\pi} - G * \sigma'_h\Big|_k - \frac{1}{L}\sum_{p=1}^{N_r}(a_2 - a_4)\Big|_{\Lambda - k_p}, \tag{58}$$

where $G = a_1 * (I + a_2)^{-1}$. The kernel of G is given by

$$G(\Lambda) = \frac{1}{2c\cosh\left(\frac{\pi\Lambda}{c}\right)}. \tag{59}$$

As a result we have

$$f(k) = \frac{k}{2\pi} - \frac{1}{2\pi L}\sum_p \theta\Big(\frac{k-k_p}{c}\Big) - \theta\Big(\frac{k-k_p}{2c}\Big) - \Gamma * \sigma'_h\Big|_k + \chi, \tag{60}$$

where the k-independent piece χ is fixed by the requirement that $f(\infty) = 0$, and Γ is an integral operator with the kernel

$$\Gamma(x) = \frac{1}{\pi}\arctan\Big(\tanh\Big(\frac{\pi x}{2c}\Big)\Big). \tag{61}$$

In order to proceed we need to evaluate $\Gamma * \sigma'_h$. For the ground state this is simply given by

$$\Gamma * \sigma'_h\Big|_k = \int_{Q_0}^{\infty} d\Lambda\,[\Gamma(\Lambda + k) - \Gamma(\Lambda - k)]\,\sigma'_0(\Lambda)\ , \tag{62}$$

where

$$\sigma'_0(\Lambda) = \frac{1}{\pi} + \int_{-Q_0}^{Q_0} d\Lambda'\ K(\Lambda - \Lambda')\ \sigma'_0(\Lambda'). \tag{63}$$

Here we have defined a kernel

$$K = a_6 + a_4 - a_2. \tag{64}$$

Using the relations $\sigma_0(\Lambda) = \sigma_0(-\Lambda)$, $\lim_{\Lambda\to\infty}\sigma_0(\Lambda) = \frac{1}{\pi}$ and taking into account that by virtue of Eq. (62) the quantity $\Gamma * \sigma'_h\Big|_k$ is an odd function of k we conclude that for the ground state configuration we have $\chi = 0$. Let us now consider the limit where

$$c, |k| \ll Q_0. \tag{65}$$

For large x we have

$$\Gamma(x) \longrightarrow \frac{1}{4} - \frac{1}{\pi}e^{-\pi x/c}\ , \quad x \gg c, \tag{66}$$

which yields

$$\Gamma * \sigma'_h\Big|_k \approx \frac{2}{\pi}\sinh\Big(\frac{\pi k}{c}\Big)\int_{Q_0}^{\infty} d\Lambda e^{-\pi\Lambda/c}\ \sigma'_0(\Lambda) \equiv \frac{\Delta}{2\pi}\sinh\Big(\frac{\pi k}{c}\Big). \tag{67}$$

For an excited state with n_p particles and n_h holes with corresponding spectral parameters $\Lambda_j^{p,h}$ added to the ground state distribution of k-Λ strings as well as N_r additional real k's, the integral equation for the total root density $\sigma' = \sigma_p' + \sigma_h'$ reads

$$\sigma'(\Lambda) = \frac{1}{\pi} + \int_{-Q}^{Q} d\Lambda' \; K(\Lambda - \Lambda') \; \sigma'(\Lambda') \\ + \frac{1}{L}\left[\sum_{l=1}^{n_p} K(\Lambda - \Lambda_l^p) - \sum_{r=1}^{n_h} K(\Lambda - \Lambda_r^h) + \sum_{j=1}^{N_r} (a_5 - a_1)(\Lambda - k_j)\right]. \qquad (68)$$

Here the integration boundary Q may differ from the one for the ground state (Q_0) by a contribution of at most order $\mathcal{O}(L^{-1})$. The integral equation (68) can be solved by an expansion in inverse powers of L

$$\sigma'(\Lambda) = \sigma_0'(\Lambda) + \frac{1}{L}\sigma_1'(\Lambda) + \mathcal{O}(L^{-2}), \qquad (69)$$

where σ_0 is the root density or the ground state (62). Hence the leading contribution (in $1/L$) to $\Gamma * \sigma_h'\Big|_k$ is given by (67). The contribution due to σ_1' can be seen to be of the form

$$\frac{1}{L}\left[\text{const} + \mathcal{O}\Big(\exp\left[-\frac{\pi Q}{c}\right]\Big)\right], \quad c, |k| \ll Q. \qquad (70)$$

The constant (i.e. independent of k) contribution is by construction precisely cancelled by χ in (60), so that

$$f(k) = \frac{k}{2\pi} - \frac{\Delta}{2\pi}\sinh\Big(\frac{\pi k}{c}\Big) - \frac{1}{2\pi L}\sum_{p=1}^{N_r} \theta\Big(\frac{k-k_p}{c}\Big) - \theta\Big(\frac{k-k_p}{2c}\Big) \\ + \frac{1}{L}\mathcal{O}\Big(\exp\left[-\frac{\pi Q}{c}\right]\Big). \qquad (71)$$

Substituting this into (60), (56) and dropping the exponentially small (in Q/c) contribution leads to

$$\prod_{p=1}^{N_b} \mathcal{P}(k - \Lambda_p) = e^{iLk}\; e^{-iL\Delta\sinh(\pi k/c)} \prod_{p=1}^{N_r} e_{-4}(k-k_p) e_2(k-k_p). \qquad (72)$$

Using this in the Bethe ansatz equations (41) and then rescaling variables by

$$k = \frac{c\theta}{\pi}\;, \quad \lambda = \frac{cu}{\pi}, \qquad (73)$$

we arrive at the Bethe ansatz equations for the O(3) nonlinear sigma model [20, 21]:

$$\mathrm{e}^{\mathrm{i}\Delta \mathrm{L} \sinh\theta_\mathrm{l}} = \prod_{\mathrm{j}=1}^{\mathrm{N_r}} \frac{\theta_\mathrm{l} - \theta_\mathrm{j} + \mathrm{i}\pi}{\theta_\mathrm{l} - \theta_\mathrm{j} - \mathrm{i}\pi} \prod_{\mathrm{a}=1}^{\mathrm{M_r}} \frac{\theta_\mathrm{l} - \mathrm{u_a} - \mathrm{i}\pi}{\theta_\mathrm{l} - \mathrm{u_a} + \mathrm{i}\pi}\;, \quad \mathrm{l} = 1, \ldots \mathrm{N_r}, \qquad (74)$$

$$\prod_{j=1}^{N_r} \frac{u_a - \theta_j + \mathrm{i}\pi}{u_a - \theta_j - \mathrm{i}\pi} = -\prod_{b=1}^{M_r} \frac{u_a - u_b + \mathrm{i}\pi}{u_a - u_b - \mathrm{i}\pi}\;, \quad a = 1, \ldots, M_r. \qquad (75)$$

The contribution of spin excitations to the total momentum is given by

$$P_{\rm spin} = \Delta \sum_{l=1}^{N_r} \sinh\theta_l. \tag{76}$$

We note that in addition to (75) we still have "physical" Bethe ansatz equations that determine the rapidities of the particle and hole excitations over the sea of k-Λ strings forming the ground state. It follows from the finite-volume quantization conditions (75) that the dressed phase-shifts for scattering of such particles and holes with excitations in the spin sector vanish (up to exponentially small corrections in Q/c). Hence the quantization conditions in the gapless sector must be of the form

$$\begin{aligned} e^{iLP(\Lambda_j^p)} &= \prod_{k\neq j}^{n_p} S_{pp}(\Lambda_j^p - \Lambda_k^p) \prod_{m=1}^{n_h} S_{ph}(\Lambda_j^p - \Lambda_m^h)\ ,\ j=1,\dots,n_p\ ,\\ e^{iLP(\Lambda_l^h)} &= \prod_{k=1}^{n_p} S_{hp}(\Lambda_l^h - \Lambda_k^p) \prod_{m\neq l}^{n_h} S_{hh}(\Lambda_l^h - \Lambda_m^h)\ ,\ l=1,\dots,n_h. \end{aligned} \tag{77}$$

Here $P(\Lambda)$ is the dressed momentum for k-Λ strings and $S_{ab}(\Lambda)$ the dressed scattering phases for particles and holes. They can be expressed in terms of the solution to the integral equation (68) following standard methods [16, 22].

3.1. Dressed Energies

We will now show that the energy in the spin sector is given by

$$E_{\rm spin} = \Delta v_c \sum_{l=1}^{N_r} \cosh\theta_l\ . \tag{78}$$

Our starting point are the equations for the dressed energies for k-Λ strings $\epsilon'(\Lambda)$ and real k's $\kappa(k)$ [12]

$$\begin{aligned} \epsilon'(\Lambda) &= \frac{\Lambda^2}{m} - \frac{c^2}{4m} - 2\mu + \int_{-Q_0}^{Q_0} d\Lambda'\ K(\Lambda-\Lambda')\ \epsilon'(\Lambda')\ ,\\ \kappa(k)\ &= \frac{k^2}{2m} - \mu + \int_{-Q_0}^{Q_0} d\Lambda\ (a_5 - a_1)(k-\Lambda)\ \epsilon'(\Lambda). \end{aligned} \tag{79}$$

Here μ is the chemical potential related to Q_0 by the requirement

$$\epsilon'(Q_0) = 0. \tag{80}$$

Using the integral equation for ϵ' in the equation for κ we find that

$$\kappa(k) = \int_{Q_0}^{\infty} d\Lambda\, [G(k-\Lambda) + G(k+\Lambda)]\, \epsilon'(\Lambda), \tag{81}$$

where G is given in Eq. (59). In the limit $c, |k| \ll Q_0$ this simplifies to

$$\kappa(k) \approx \cosh\Big(\frac{\pi k}{c}\Big)\ \frac{2}{c}\int_{Q_0}^{\infty} d\Lambda\ e^{-\pi\Lambda/c}\ \epsilon'(\Lambda). \tag{82}$$

Expanding ϵ' around Q_0 we have

$$\epsilon'(\Lambda) = \frac{d\epsilon'}{d\Lambda}\Big|_{Q_0} (\Lambda - Q_0) + \dots \tag{83}$$

Noting that the Fermi velocity in the charge sector is defined as

$$v_c = \frac{d\epsilon'}{dp}\Big|_{Q_0} = \frac{\frac{d\epsilon'}{d\Lambda}}{\frac{dp}{d\Lambda}}\Big|_{Q_0}, \tag{84}$$

where $p(\Lambda)$ is the dressed momentum for k-Λ strings and using that this may be expressed as $p(\Lambda) = y(\Lambda)$ by virtue of Eqs. (46) and (45), we conclude that

$$v_c = \frac{\frac{d\epsilon'}{d\Lambda}}{2\pi\sigma'(\Lambda)}\Big|_{Q_0}, \tag{85}$$

and thus

$$\kappa(k) \approx v_c \cosh\Big(\frac{\pi k}{c}\Big)\ \frac{4c}{\pi}\sigma'(Q_0)\ e^{-\pi Q_0/c}. \tag{86}$$

Evaluating the right-hand side of Eq. (67) by using an analogous approximation and rescaling k as before we conclude that indeed

$$\kappa(\theta) = \Delta v_c \cosh\theta. \tag{87}$$

4. Scaling Dimensions.

The analysis of the finite-size spectrum is complicated by the fact that the ground state is made of strings. A priori this could lead to complications related to deviations of the strings from their ideal forms (see, e.g. [23]). However, as the critical sector of our theory is a simple compactified boson we do not expect such complications to play a role. The dressed charge $Z = Z(Q_0)$ is defined in terms of the solution of the integral equation [24]

$$Z(\Lambda) = 1 + \int_{-Q_0}^{Q_0} d\Lambda'\ K(\Lambda - \Lambda')\ Z(\Lambda'). \tag{88}$$

We see that we have

$$Z = \pi\sigma'(Q_0). \tag{89}$$

It is easy to see that the dressed charge only depends on the dimensionless ratio

$$\gamma = \frac{c}{\rho_0}. \tag{90}$$

At small γ we find that

$$Z \to \frac{\sqrt{\pi}}{2}\gamma^{-1/4} \gg 1. \tag{91}$$

On the other hand, for $\gamma \to \infty$ the dressed charge approaches 1 from below. This implies that Z is not a monotonic function of γ, but instead has a local minimum. The

finite size spectra of energy and momentum are expressed in terms of the dressed charge as [25]

$$E(\Delta N_b, D, N_+, N_-) = \frac{2\pi v_c}{L}\left[\frac{(\Delta N_b)^2}{4Z^2} + Z^2 D^2 + N_+ + N_-\right], \tag{92}$$

$$P(\Delta N_b, D, N_+, N_-) = \frac{2\pi}{L}\left[N_+ - N_- + D(\Delta N_b)\right] + 2k_F D. \tag{93}$$

Here k_F is given by Eq. (48), ΔN_b is the change in bound state number compared to the ground state and D is an integer. The corresponding spectrum of scaling dimensions is given by

$$\begin{aligned} h(\Delta N_b, D, N_+, N_-) &= \frac{1}{2}\left[DZ + \frac{(\Delta N_b)}{2Z}\right]^2 + N_+ , \\ \bar{h}(\Delta N_b, D, N_+, N_-) &= \frac{1}{2}\left[DZ - \frac{(\Delta N_b)}{2Z}\right]^2 + N_-. \end{aligned} \tag{94}$$

Let us look at some specific examples. The lowest charge-neutral excitations with momenta $2k_F$ and 0 respectively correspond to $\Delta N_b = N_\pm = 0$, $D = 1$ and $D = N_b = N_- = 0$, $N_+ = 1$

$$\begin{aligned} h(0,1,0,0) &= \frac{Z^2}{2} , \\ h(0,0,1,0) &= 1. \end{aligned} \tag{95}$$

The lowest excited state with charge 2 corresponds to $\Delta N_b = 1$, $N_\pm = D = 0$

$$h(1,0,0,0) = \frac{1}{8Z^2}. \tag{96}$$

5. Correlation functions.

Let us now turn to the large-distance asymptotics of correlation functions. We start by considering the gapless sector of the low-energy effective action, which is given by

$$S_0 = \frac{K_c}{2\pi}\int dt dx \left[\frac{1}{v_c}(\partial_t\phi_+)^2 - v_c(\partial_x\phi_+)^2\right]. \tag{97}$$

Given that the original phase fields $\phi_{1,2}$ were 2π-periodic, the field ϕ_+ is in fact π-periodic (see Eq. (8)). As a result, the spectrum of scaling dimensions of primary operators in the Gaussian model (97) is given by

$$h(m,n) = \frac{1}{8}\left(\frac{2m}{\sqrt{K}} + n\sqrt{K}\right)^2 , \qquad \bar{h}(m,n) = \frac{1}{8}\left(\frac{2m}{\sqrt{K}} - n\sqrt{K}\right)^2 , \tag{98}$$

where n and m are integers. We note that (98) agrees with (94) in the appropriate limit, where we have $\sqrt{K} = 2Z$. The operators corresponding to these scaling dimensions are constructed by introducing the field dual to ϕ_+ by [26]

$$[\theta_+(x), \phi_+(x')] = i\frac{\pi}{2}\mathrm{sgn}(x - x'). \tag{99}$$

The boson number is given by

$$N = \frac{1}{\pi}\int dx\ \partial_x\theta_+(x), \tag{100}$$

The local operators corresponding to the spectrum of scaling dimensions (98) are

$$\mathcal{O}_{2m,n} = e^{2im\phi_+ + in\theta_+}\ ;\quad m, n \text{ integer.} \tag{101}$$

These operators carry charge $2m$, reflecting the fact that our fundamental objects are pairs of bosons and have two-point functions of the form‡

$$\langle \mathcal{O}_{2m,n}(x)\mathcal{O}^\dagger_{2m,n}(0,0)\rangle = \left[\frac{a_0}{x}\right]^{2h(m,n)+2\bar{h}(m,n)} e^{in\pi\rho_0 x}, \tag{102}$$

where a_0 is a short-distance cutoff determined by the gapped degrees of freedom we have integrated out in order to arrive at the action (22), (24). We are now in a position to write down expressions for operators that do not involve the spin sector. These are the pairing field Δ and the density, which are given by

$$\begin{aligned} \Delta\ \ &= \Psi_1\Psi_{-1} - \Psi_0^2/2\ , \\ :\rho:\ &= \Psi^+_\sigma\Psi_\sigma - \rho_0\ . \end{aligned} \tag{103}$$

As Δ carries charge -2 we conclude that it takes the form

$$\Delta(x) = \rho_0 \sum_{n=-\infty}^{\infty} A_n \mathcal{O}_{2,n}(x)\ , \tag{104}$$

where A_n are numerical coefficients. We will see below that $A_{2k+1} = 0$. Similarly, as the density is neutral it is given by

$$:\rho:= \frac{1}{\pi}\partial_x\theta_+ + \sum_{n=-\infty}^{\infty} B_n\mathcal{O}_{0,n}(x)\ . \tag{105}$$

Like for the pairing field, it turns out that by virtue of the statistics of the Bose field all odd coefficients vanish, i.e. $B_{2k+1} = 0$. Let us now turn to the Bose field itself. At low energies the semiclassical analysis gives

$$\Psi_\sigma(x) \sim \sqrt{\rho_0}\mathrm{e}^{\mathrm{i}\phi_+}\begin{pmatrix} \frac{m_1+im_2}{\sqrt{2}} \\ m_3 \\ -\frac{m_1-im_2}{\sqrt{2}} \end{pmatrix}. \tag{106}$$

Since the spin part of the bosonic operator is proportional to the slowly varying unit vector field $\mathbf{m}$, we conclude that the triplet excitations of the nonlinear sigma model occur at small momenta $q \approx 0$. To take into account the quantization of total charge we should complete Eq. (106) with a piece containing the field dual to ϕ_+, which amounts to the substitution

$$\sqrt{\rho_0}\mathrm{e}^{i\phi_+} \longrightarrow \sqrt{\rho_0 + \frac{1}{\pi}\partial_x\theta_+}\sum_n C_{2n} e^{2in\theta_+ + 2in\pi\rho_0 x}\mathrm{e}^{\mathrm{i}\phi_+}\ . \tag{107}$$

We note that only even exponentials of the dual field are allowed by virtue of the bosonic statistics of the fields Ψ_σ, which requires the Lorentz spin of the operators appearing in Eq. (107) to be integer. At $T = 0$ we then obtain

$$\langle T_\tau \Psi^\dagger_\sigma(\tau,x)\Psi_{\sigma'}(0,0)\rangle = \delta_{\sigma,\sigma'}\ \langle T_\tau m_\sigma(\tau,x)\ m_\sigma(0,0)\rangle f_{\rm charge}(\tau,x)\ , \tag{108}$$

‡ We note that if ϕ_+ were a 2π-periodic field, the allowed scaling dimensions would be $h_{2\pi}(m,n) = \frac{1}{8}\left(\frac{m}{\sqrt{K}} + 2n\sqrt{K}\right)^2$, $\bar{h}_{2\pi}(m,n) = \frac{1}{8}\left(\frac{m}{\sqrt{K}} - 2n\sqrt{K}\right)^2$ and correspond to operators $\mathcal{O}_{m,2n}$. This is the case for Haldane's "bosonization of the boson" [26].

$$f_{\rm charge}(\tau,x)=\rho_0\left[\frac{a_0^2}{(v^2\tau^2+x^2)}\right]^{1/4K_c}$$
$$+\rho_0\left[\frac{a_0^2}{(v^2\tau^2+x^2)}\right]^{K_c+1/4K_c}\left[C_2^2e^{2\pi i\rho_0 x}\frac{v\tau+ix}{v\tau-ix}+\text{h.c.}\right]+\dots. \tag{109}$$

The correlation function of the sigma model is well approximated by the expression [27]

$$\langle T_\tau m_\sigma(\tau,x)\ m_\sigma(0,0)\rangle\approx ZK_0(\Delta\sqrt{\tau^2+x^2/v^2}), \tag{110}$$

where Z is a normalization. We note that finite temperature correlation functions can be calculated by using the conformal mapping in the gapless sector and the method of Ref. [28] for the nonlinear sigma model.

The expression for the Bose-fields feed back to the low-energy projections of the density and pairing field, which are bilinears in the Bose fields. Given that only even bosonic exponentials of the dual field appear in the expression for Ψ_σ, the same must hold for $:\rho:$ and Δ. This requirement sets the coefficients of $\mathcal{O}_{2n,2k+1}$ in the expressions for $:\rho:$ and Δ to zero. The two-point functions of both these operators undergo a power law decay at large distances

$$\langle\Delta(x)\Delta^\dagger(0)\rangle\sim\rho_0^2|A_0|^2\left[\frac{a_0^2}{x^2}\right]^{1/K_c}$$
$$+2|A_2|^2\cos(2\pi\rho_0 x)\left[\frac{a_0^2}{x^2}\right]^{K_c+1/K_c}+\dots\ , \tag{111}$$

$$\langle:\rho(x)::\rho(0):\rangle\sim\frac{K_c}{2\pi^2x^2}+2B_2^2\cos(2\pi\rho_0x)\left[\frac{a_0^2}{x^2}\right]^{K_c}+\dots. \tag{112}$$

We see that the oscillating piece of the two-point function of the pairing field Δ always decays more rapidly with distance than the oscillating piece of the charge density.

5.1. Optical Lattice

The model (2) arises as the low-density continuum limit of a three-component Bose-Hubbard system

$$H_{BH}=-t\sum_\sigma a_{j,\sigma}^\dagger a_{j+1,\sigma}+\text{h.c.}+U_0\sum_j n_j^2+U_1\sum_j\boldsymbol{S}_j^2-\mu\sum_j n_j\ , \tag{113}$$

where $n_j=\sum_\sigma a_{j,\sigma}^\dagger a_{j,\sigma}$ and S_j^α are spin-1 operators on site j. An interesting question is what the phase diagram of the model (113) looks like. A Density Matrix Renormalization Group analysis [4] has established the regions in parameter space where Mott phases occur. In order to address this issue by analytical methods one would need to bosonize the Hamiltonian (113) for strong interactions and large densities. Assuming that the operator content suggested by Eqs. (106) and (107) remains unchanged, this would occur when the oscillating part of the Bose field becomes sufficiently relevant so that the oscillating parts of the operator n_j^2 turn into a relevant perturbation in the entire charge sector. The perturbation is of the form

$$\delta S=\lambda\int dtdx\ \mathcal{O}_{0,4}(t,x). \tag{114}$$

It is relevant in the RG sense if $K_c < \frac{1}{2}$. It would be interesting to check whether the phase boundaries established in [4] coincide with the loci in parameter space where K_c becomes 1/2.

6. Discussion.

In this work we have analyzed a continuum model of a spinor Bose gas in one spatial dimension. Using standard semiclassical methods we showed that for weak interactions the low energy degrees of freedom are described by a spin-charge separated theory of a free boson and the O(3) nonlinear sigma model. We then turned to the integrable line in the model. From the exact Bethe ansatz solution we obtained an independent derivation of the low-energy effective theory. An interesting by-product of our analysis is the demonstration that the integrable model provides a simple, integrable regularization of the O(3) nonlinear sigma model. Finally, we determined the long-distance asymptotics of certain correlation functions (including the "order-parameter" correlator) in the framework of the low-energy description. The dominant fluctuations in the theory are of superfluid singlet pair type. Our analysis for dynamical correlations pertains mostly to the weak-coupling regime. The opposite limit of (infinitely) strong repulsive density-density interaction should amenable to a treatment along the lines set out in [29]. Other interesting open questions are how the 1D spinor Bose gas behaves under a quantum quench [30] and what happens in the unbalanced case where the densities of the three species of bosons are fixed at different values. Finally, it would be interesting to generalize our analysis to spin S bosons [31].

Acknowledgments

We thank D. Kovrizhin, P. Lecheminant, J.E. Moore and particularly D. Schuricht for helpful discussions. The work was supported by the EPSRC under grant EP/D050952/1 (FHLE), the US DOE under contract number DE-AC02-98 CH 10886 (AMT), the IFRAF Institute (AMT), the ANR through grants 05-BLAN-0205 and 06-NANO-014-01 (GS), by the ESF networks INSTANS and QUDEDIS, and by the Dutch Foundation FOM (GS). FHLE and AMT thank the Galileo Galilei Institute for Theoretical Physics for kind hospitality and INFN for partial support during the completion of this work.

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